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USE OF WATSON OPERATIONS IN THE SOLUTION OF SOME PROBLEMS IN THE THEORY OF THERMAL CONDUCTIVITY

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UDC 517.947.43

The Watson transform method is used to obtain analytical solutions of certain nonstationary problems in the theory of thermal conductivity for variable regions with a specified law of boundary motion.

We will consider the following problem: we must find a solution of the thermal conductivity equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

in the region $l_1 + bt \le x \le l_2 + bt$, $l_2 - l_1 = l > 0$, $-\infty < t < \infty$, which satisfies the initial condition

$$u|_{t=-\infty} = 0 \tag{2}$$

and the boundary conditions

$$\left.\frac{\partial u}{\partial x} + \alpha u\right|_{x=l_1+bt} = h_1(t), \tag{3}$$

$$u|_{x=l_2+bt} = h_2(t). \tag{4}$$

Here a, b, l_1, l_2, α are constant parameters. We will seek the solution of Eqs. (1)-(4) in the form of the sum of thermal potentials of a simple and twin layer [1, 2]

$$u(x,\ t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\rho_1(s)}{\sqrt{t-s}} e^{-\frac{(x-l_1-bs)^2}{4a^2(t-s)}} ds + \frac{1}{4a\sqrt{\pi}} \int_{-\infty}^{t} \rho_2(s) \frac{(x-l_2-bs)}{(t-s)^{3/2}} e^{-\frac{(x-l_2-bs)^2}{4a^2(t-s)}} ds, \tag{5}$$

where $\rho(t)$ and $\rho_2(t)$ are unknown functions, to be defined from boundary conditions (3), (4); initial condition (2) is satisfied automatically. To define $\rho_1(t)$ and $\rho_2(t)$ we obtain a system of Voltaire integral equations of the second sort

$$h_i(t) = (-1)^i \frac{\rho_i(t)}{2} + \sum_{j=1}^{2} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s)\, \rho_j(s)\, ds \quad (i = 1,\ 2), \tag{6}$$

where

$$K_{11}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{b^2}{4a^2} x\right),$$

$$K_{12}(x) = \left(\frac{a^2 + lb - a^2\alpha l}{a^2 x^{3/2}} - \frac{l^2}{2a^2 x^{5/2}} - \frac{b^2 + 2\alpha a^2 b}{2a^2\sqrt{x}}\right) \exp\left(-\frac{b^2 x}{4a^2} + \frac{lb}{4a^2} - \frac{l^2}{4a^2 x}\right),$$

$$K_{21}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{b^2 x}{4a^2} - \frac{lb}{4a^2} - \frac{l^2}{4a^2 x}\right),$$

Computation Center, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 37, No. 3, pp. 503-507, September, 1979. Original article submitted January 29, 1979.

 0022-0841/79/3703-1094 $07.50

$$K_{22}(x) = \frac{1}{\sqrt{x}} \exp\left(-\frac{b^2 x}{4a^2}\right),$$

$$c_{11} = \frac{2a^2\alpha - b}{4a\sqrt{\pi}},$$

$$c_{12} = \frac{1}{4a\sqrt{\pi}},$$

$$c_{21} = \frac{a}{2\sqrt{\pi}},$$

$$a_{22} = \frac{b^2}{4a\sqrt{\pi}}.$$

We transform Eq. (6) to a system of integral equations with integrands dependent on the products of the arguments. With the aid of the new variables

$$t = \ln\tau, \ s = -\ln\sigma, \tag{7}$$

system (6) is reduced to the form

$$h_i(\ln\tau) = (-1)^i \frac{\rho_i(\ln\tau)}{2} + \sum_{j=1}^{2} c_{ij} \int_{1/\tau}^{\infty} K_{ij}(\ln(\tau\sigma)) \frac{\rho_i(-\ln\sigma)}{\sigma} \, d\sigma \ (i=1,\ 2). \tag{8}$$

We introduce the notation

$$\varphi_i(\tau) = \frac{\rho_i(-\ln\tau)}{\tau}, \ g_i(\tau) = 2h_i(\ln\tau) \ (i = 1,\ 2). \tag{9}$$

Then from Eq. (8) we obtain

$$g_i(\tau) = (-1)^i \frac{1}{\tau} \varphi_i\left(\frac{1}{\tau}\right) + 2\sum_{j=1}^{2} c_{ij} \int_{1/\tau}^{\infty} K_{ij}(\ln(\tau\sigma)) \, \varphi_j(\sigma) \, d\sigma \ (i = 1,\ 2). \tag{10}$$

We define the functions $\tilde{K}_{ij}(x)$, $(i,\ j = 1,\ 2)$ with the expressions

$$\tilde{K}_{ij}(x) = \frac{1}{x} \int_1^x K_{ij}(\ln t) \, dt. \tag{11}$$

We will assume that $b^2/4a^2 > 1/2$; then

$$\tilde{K}_{ij}(x) \in L_2(1;\ \infty) \quad (i,\ j = 1,2),$$

and consequently, we can solve the system of integral equations (10) by the method based on expansion of the integral operators in orthogonal Watson operators [3].

We expand $\tilde{K}_{ij}(x)$ $(i,\ j = 1,\ 2)$ over the interval $1 \le x < \infty$ in functions $x^{-1}L_n(\ln x)$ $(n = 0,\ 1,\ 2, \ldots)$, which form a complete orthonormalized system in $L_2(1,\ \infty)$. We then have

$$\tilde{K}_{ij}(x) = \sum_{n=0}^{\infty} (-1)^n a_n^{ij} x^{-1} L_n(\ln x), \tag{12}$$

where

$$a_n^{ij} = (-1)^n \int_1^{\infty} \tilde{K}_{ij}(x) \, x^{-1} L_n(\ln x) \, dx \ (i,\ j=1,\ 2;\ n=0,\ 1,\ 2,\ \ldots); \tag{13}$$

$L_n(\ln x)$ are Laguerre polynomials, defined by the expression

$$L_n(z) = \sum_{k=0}^{n} \frac{(-1)^k n!\ z^k}{(n-k)!\ (k!)^2}. \tag{14}$$

For the case where $b^2 = 4a^2$, the problem of calculating the expansion coefficients becomes simpler. For example, in this case

$$\tilde{K}_{11}(x) = \tilde{K}_{22}(x) = \frac{2\sqrt{\ln x}}{x}, \tag{15}$$

and the coefficients a_n^{11} and a_n^{22} are calculated with the expression

$$a_n^{11} = a_n^{22} = (-1)^n \, 2 \int_1^{\infty} \sqrt{\ln x}\, L_n(\ln x)\, x^{-2} dx. \tag{16}$$

After calculating the integral on the right-hand side of the last equation, we obtain

$$a_n^{11} = a_n^{22} = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k n! \, (2k+1)!!}{(n-k)! \, (k!)^2 \, 2^k}. \tag{17}$$

In the general case, the coefficients a_n^{ij} ($i, j = 1, 2; n = 0, 1, 2, \ldots$) are defined by some numerical method. Considering that

$$\int_{1/\tau}^{\infty} K_{ij}(\ln(\sigma\tau))\, \varphi_j(\sigma)\, d\sigma = \frac{d}{d\tau}\left\{\tau \int_{1/\tau}^{\infty} \tilde{K}_{ij}(\tau\sigma)\, \varphi_j(\sigma)\, d\sigma\right\}, \tag{18}$$

with consideration of Eq. (12) we obtain

$$\int_{1/\tau}^{\infty} K_{ij}(\ln(\sigma\tau))\, \varphi_j(\sigma)\, d\sigma = \sum_{n=0}^{\infty} a_n^{ij} \frac{d}{d\tau}\left\{\tau \int_{1/\tau}^{\infty} (-1)^n (\sigma\tau)^{-1} L_n(\ln(\sigma\tau))\, \varphi_j(\sigma)\, d\sigma\right\}. \tag{19}$$

The functions $x^{-1}L_n(\ln x)$ are the integrands of Watson operators $(-1)^n S(TS)^n$; $n = 0, 1, 2, \ldots$ Therefore, from Eq. (19) we obtain

$$\int_{1/\tau}^{\infty} K_{ij}(\ln(\sigma\tau))\, \varphi_j(\sigma)\, d\sigma = \sum_{n=0}^{\infty} a_n^{ij} S(TS)^n \varphi_j(\tau), \tag{20}$$

where S and T are elementary Watson operators.

With the aid of Eq. (20) the system of integral equations (10) can be written in the following operator form:

$$(-1)^i S\varphi_i(\tau) + 2\sum_{j=1}^{2} c_{ij} \sum_{n=0}^{\infty} a_n^{ij} S(TS)^n \varphi_j(\tau) = g_i(\tau) \quad (i = 1, 2). \tag{21}$$

Applying the operator S to both sides of Eq. (21) and introducing the notation

$$Z_{11} = 2c_{11} \sum_{n=0}^{\infty} a_n^{11} (TS)^n - E, \tag{22}$$

$$Z_{12} = 2c_{12} \sum_{n=0}^{\infty} a_n^{12} (TS)^n,$$

$$Z_{21} = 2c_{21} \sum_{n=0}^{\infty} a_n^{21} (TS)^n,$$

$$Z_{22} = 2c_{22} \sum_{n=0}^{\infty} a_n^{22} (TS)^n + E,$$

where E is the identical conversion operator, we obtain

$$\sum_{k=1}^{2} Z_{ik}\varphi_k(\tau) = Sg_i(\tau) \quad (i = 1, 2). \tag{23}$$

Let Δ be the operator determinant of system (23); we denote the operators which are formally the algebraic complements of the elements of this determinant by the symbols A_{ik}. The operators A_{ik} are the sums of products of operators of the form of Eq. (22) and can be expressed as series in nonorthogonal powers of the operator TS. Considering the existence of an inverse operator Δ^{-1}, the solution of Eq. (23) may be written in the form

$$\varphi_k(\tau) = \Delta^{-1} \sum_{i=1}^{2} A_{ki} S g_i(\tau) \quad (k = 1, 2). \tag{24}$$

The desired functions $\rho_1(t)$ and $\rho_2(t)$ may now be expressed with the formulas

$$\rho_k(t) = e^{-t}\varphi_k(e^{-t})\ (k = 1,\ 2).$$

The solution of the original problem can be represented in the form

$$u(x,\ t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\varphi_1(e^{-s})}{\sqrt{t-s}} \exp\left(-\frac{(x-l_1-bs)^2}{4a^2(t-s)} - s\right) ds$$
$$+ \frac{1}{4a\sqrt{\pi}} \int_{-\infty}^{t} \frac{\varphi_2(e^{-s})}{(t-s)^{3/2}} (x-l_2-bs) \exp\left(-\frac{(x-l_2-bs)^2}{4a^2(t-s)} - s\right) ds. \tag{25}$$

In particular, let $l_2 = +\infty$, which corresponds to the solution of thermal conductivity equation (1) with one boundary condition (3) in the region $x > l_1 + bt$, $-\infty < t < \infty$ with initial condition (2). Then $\rho_2(t) = h_2(t) = 0$ and in system (23) there remains one equation

$$Z_{11}\varphi_1(\tau) = Sg_1(\tau). \tag{26}$$

The solution of operator equation (26) has the form

$$\varphi_1(\tau) = 2Z_{11}^{-1}Sh_1(\ln\tau).$$

The desired function $\rho_1(t)$ is given by the expression

$$\rho_1(t) = e^{-t}\varphi_1(e^{-t}).$$

The solution of the original thermal conductivity problem in the special case considered ($l_2 = +\infty$) has the form

$$u(x,\ t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{\varphi_1(e^{-s})}{\sqrt{t-s}} \exp\left(-\frac{(x-l_1-bs)^2}{4a^2(t-s)} - s\right) ds. \tag{27}$$

Now let $l_1 = -\infty$. In this case we consider thermal conductivity equation (1) with one boundary condition (4) in the region $x \leq l_2 + bt$, $-\infty < t < \infty$ with initial condition (2). Then $\rho_1(t) = h_1(t) = 0$ and again in system (23) there remains one equation

$$Z_{22}\varphi_2(\tau) = Sg_2(\tau), \tag{28}$$

whence

$$\varphi_2(\tau) = 2Z_{22}^{-1}Sh_2(\ln\tau);$$

and the desired function $\rho_2(t)$ is given by

$$\rho_2(t) = e^{-t}\varphi_2(e^{-t}).$$

In this case ($l_1 = -\infty$) the solution of the original thermal conductivity problem has the form

$$u(x,\ t) = \frac{1}{4a\sqrt{\pi}} \int_{-\infty}^{t} \frac{\varphi_2(e^{-s})}{(t-s)^{3/2}} (x-l_2-bs) \exp\left(-\frac{(x-l_2-bs)^2}{4a^2(t-s)} - s\right) ds. \tag{29}$$

A detailed study of one special case of system (10) was presented by Shub-Sizenenko [4].

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